Algebraic Geometry Final Exam

April 30 2026

This exam is of **50 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. You may consult the books of Miles Reid - '**Undergraduate Algebraic Geometry**' and W. Fulton - '**Algebraic Curves**'.

1. Let p be an **odd prime** number.
   1. Under what conditions on p does the equation
   $$C_p : X^2 + Y^2 = p$$
   have a rational solution? (3)
   2. Construct a birational map from $\mathbb{P}^1 \to C_{17}$ and use that to find all rational solutions. (2)
   3. Under what conditions on p does the equation
   $$D_p : X^2 - Y^2 = p$$
   have a rational solution? (3)
   4. Construct a birational map from $\mathbb{P}^1 \to D_{11}$ and use that to find all rational solutions. (2)

2. Let J be the ideal $J = (XY, YZ, XZ) \subset k[X, Y, Z]$.
   1. What is $V(J)$? (2)
   2. Is $V(J)$ irreducible? (2)
   3. Is $J = I(V(J))$? (2)
   4. What is $Rad(J)$? (2)
   5. Prove that J cannot be generated by 2 elements. (2)

3. Let $C \subset \mathbb{P}^3$ be given by $Q_1 \cap Q_2 \cap Q_3$ where

$$Q_1 : XZ = Y^2 \qquad Q_2 : XW = YZ \qquad Q_3 : YW = Z^2$$

1. Show that $C \neq Q_i \cap Q_j$ for any two i and j. (4)

2. Show that C is parameterised by (3)

$$\mathbb{P}^1 \longrightarrow C$$

$$[U, V] \longrightarrow [U^3, U^2V, UV^2, V^3]$$

3. What is the inverse map? (3)

4. We follow Fulton, Section 7.2. Let $C = V(F)$ be an **affine plane curve** where $F \in k[X, Y]$ is a polynomial with coefficients in a field k. Let $B \subset \mathbb{A}^3$ be the subvariety

$$B = \{(x, y, z) \in \mathbb{A}^3 | y = xz\}$$

and $\pi : B \to \mathbb{A}^2_{x,y}$,

$$\pi(x, y, z) = (x, y).$$

Identify B with $\mathbb{A}^2_{x,z}$ via the maps $\phi(x, z) = (x, xz, z)$, $\psi(x, y, z) = (x, z)$.

1. Show that C has finitely many multiple points. (2)

2. For $F(X, Y) = Y^2 - X^7$ and $C = V(F)$ find $F'(X, Z)$ where F' is the equation of the curve C' - where $C' = \psi(\overline{\pi^{-1}(C \backslash \{(0, 0)\})})$.

3. Let $C' = V(F')$. Is it non-singular? If not, find a curve C'' birational to C' which is non-singular. (3)

4. How many points are in the pre-image of $(0, 0)$ under the map $C'' \to C$? (2)

5. Let C be a **cubic plane curve**.

1. How many multiple points can C have. (2)

2. What are the possible genera (genus) that C can have? (3)

3. Under what conditions on C do there exist two distinct points P and Q such that $P \sim Q$? Justify your answer. (5)